

**Isolating the Systematic and Unsystematic Components of a Single Stock's  
(or Portfolio's) Standard Deviation**

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# **Isolating the Systematic Component of a Single Stock's (or Portfolio's) Standard Deviation**

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## **ABSTRACT**

This paper revisits the roots of modern portfolio theory and the recognition that a stock's (or a stock portfolio's) risk can be decomposed into a systematic component and an unsystematic component, and, further, that only the former should contribute to expected return. However, instead of isolating the systematic component of risk by recasting the risk in terms of a stock's beta coefficient, I choose to decompose the standard deviation, or variance if one prefers the original risk measure, directly into its systematic and unsystematic components allowing one to focus on systematic risk and yet remain in the mean/standard deviation (or mean/variance) space. When the standard deviation of return is decomposed into its systematic and unsystematic components, an "adjusted CML" can be derived and it is easily shown that this adjusted CML is equivalent to Sharpe's SML.

This alternative way of looking at systematic and unsystematic risk offers easily accessible insights into the very nature of risk. This has a number of interesting implications including, but not limited to, reducing the computational complexities in calculating the relevant portion of a portfolio's volatility, facilitating sophisticated dispersion trades, estimating risk-adjusted returns, and improving risk-adjusted performance measurement.

This paper is, in part, pedagogical and, in part, an introduction to an alternative way of measuring systematic and unsystematic risk.

**KEY WORDS:** systematic risk, unsystematic risk, capital asset pricing model, dispersion trading, risk-adjusted performance measurement

## INTRODUCTION

Parts of this paper can be viewed as purely pedagogical. As such, I have included proofs of several well-known fundamental tenets of modern portfolio theory and its extensions. These proofs have been relegated to appendices and can be skipped by those thoroughly versed in the fundamentals.

In 1952, Markowitz demonstrated that diversification reduces the risk associated with holding a portfolio of assets. This work became the foundation of all modern portfolio theory and was instrumental in the further evolution of a number of branches of financial theory including, but not limited to, capital asset pricing theory, hedging theory, option pricing theory, and asset allocation theory.

While it was intuitively understood for at least three centuries that diversification reduces risk,<sup>i</sup> until Markowitz's (1952, 1959) pioneering work, no one had derived the precise relationship between diversification and risk reduction. Markowitz showed that the key to understanding the benefits of diversification lay in three metrics: the weights employed in the assignment of wealth to individual portfolio components, the standard deviations of the individual portfolio components' returns, and the pair-wise return correlations of the various components of the portfolio. While most often presented in the context of a stock portfolio, the principles are equally relevant to any asset class and to diversification across asset classes.<sup>ii</sup> In this paper I will cast the conversation in terms of individual stocks and stock portfolios. (The reader is invited to substitute the word "security" or the word "asset" for the word "stock" to broaden the context.) I will denote the number of stocks in the portfolio by  $N$  and the number of stocks in the market by  $M$  (later I will use the S&P 500 as a proxy for the market, so  $M = 500$ ).

Markowitz showed that the total risk, as measured by the variance of return  $\sigma_p^2$ , associated with a stock portfolio is given by Equation (1). (See Appendix 1 for a formal proof.)

$$\sigma_p^2 = \sum_{i=1}^N \sum_{j=1}^N w_i w_j \sigma_{i,j} \quad (1)$$

where  $w_i$  denotes the weight on stock i,  $w_j$  denotes the weight on stock j, and  $\sigma_{i,j}$  denotes the covariance between the returns on stocks i and j. By finding the weights that minimize the variance for any given expected return, Markowitz was able to derive the minimum-variance set of which the efficient frontier is a subset.<sup>iii</sup> The efficient frontier proved to be quadratic in nature.

Recognizing that, by definition,  $\sigma_{i,j} = \sigma_i \sigma_j \rho_{i,j}$ , Equation (1) can be written as Equation (2) where  $\sigma_i$  and  $\sigma_j$  denote the standard deviations of the returns for stock i and stock j, respectively, and  $\rho_{i,j}$  denotes the return correlation of stocks i and j.

$$\sigma_p^2 = \sum_{i=1}^N \sum_{j=1}^N w_i w_j \sigma_i \sigma_j \rho_{i,j} \quad (2)$$

Segregating the cases where  $i = j$  from the cases where  $i \neq j$ , Equation (2) can be written as Equation (3):

$$\sigma_p^2 = \sum_{i=1}^N w_i^2 \sigma_i^2 + \sum_{i=1}^N \sum_{\substack{j=1 \\ i \neq j}}^N w_i w_j \sigma_i \sigma_j \rho_{i,j} \quad (3)$$

It was soon recognized that the first term on the right goes to zero as the degree of diversification increases.<sup>iv</sup> For this reason, this portion of a portfolio's overall risk was originally referred to as "diversifiable risk" but later came to be known as "unsystematic risk".<sup>v</sup> The second term on the right does not diversify away and was originally called "nondiversifiable risk" but later became known as "systematic risk".<sup>vi</sup> The proof that the unsystematic risk diversifies away is provided in Appendix 2.

As a result, for a well-diversified portfolio, the unsystematic risk can be ignored and only the systematic risk remains. Therefore, as  $N \rightarrow M$  (the number of stocks in the market proxy, assumed to be large), Equation (3) becomes Equation (4):

$$\sigma_p^2 = \sum_{i=1}^N \sum_{\substack{j=1 \\ i \neq j}}^N w_i w_j \sigma_i \sigma_j \rho_{i,j} \quad (4)$$

Importantly, in Markowitz's framework, systematic risk is represented by the portion of the portfolio's overall risk that is associated with the degree to which the returns of the various pairs of included stocks are cross correlated with one another. That is, Markowitz did not attempt to separate the systematic from the unsystematic risk for individual securities. Note also that, all other things being equal, the lower are the cross-correlations of returns, the greater is the risk-reducing benefits from diversification.

To facilitate what follows in the next section, it is useful to review the mathematics of the fundamental risk concepts that grew from Markowitz's original work.

In 1958, Tobin extended Markowitz's work by demonstrating that the inclusion of a risk-free asset results in a linearization of the efficient frontier when risk is measured via standard deviation.<sup>vii</sup> This gave rise to Tobin's Separation Theorem and the Capital Market Line or CML:

$$\mu_p = \frac{(\mu_m - r_f)}{\sigma_m} \sigma_p + r_f \quad (5)$$

where  $\mu_p$  denotes the expected return for the portfolio (i.e., the fair return),  $\mu_m$  denotes the expected return for the market,  $\sigma_m$  denotes the standard deviation of return for the market,  $\sigma_p$  denotes the standard deviation of the return for the portfolio, and  $r_f$  denotes the risk-free rate.

The CML has limited applicability in the sense that it can only be expected to work correctly when it is used to estimate the expected return associated with a linear combination of the risk-free asset and the market portfolio. Nevertheless, in an efficient market, the CML would be expected to generate an unbiased estimate of the expected return for any thoroughly-diversified portfolio.<sup>viii</sup> When applied to a portfolio that is less than thoroughly diversified, the CML will overestimate the portfolio's expected return because its mathematical structure implies compensation for all portfolio risk, including unsystematic risk.<sup>ix</sup>

But how was one to isolate the systematic component of a single stock's overall risk? The breakthrough came in the early 1960s with the introduction of various versions of what are now known as the Capital Asset Pricing Model (CAPM), the most often employed version of which is the Security Market Line (SML) attributable to Sharpe.<sup>x</sup>

Sharpe argued that the systematic risk of a stock should be measured as the ratio of (1) the covariance of that stock's excess return with the broad market's excess return to (2) the variance of the broad market's excess return. He named this measure "beta." However, when

the risk-free rate can be viewed as constant, as it can in a forward-looking model, Sharpe's beta can be shown to be equivalent to the ratio of (1) the covariance of the raw return of the stock with the raw return of the market to (2) the variance of the raw return of the market. The proof of this proposition is provided in Appendix 3. This proof ties Sharpe's SML version of the CAPM to his earlier Single Index Model<sup>xi</sup> or SIM (also known as the Market Model) version of the CAPM, which uses raw returns. For this reason, most estimates of beta are based on raw returns rather than excess returns, and I adopt this convention.

Much has been written on the influence of the differencing interval employed in the computation of beta<sup>xii</sup>. Ignoring issues associated with the choice of the differencing interval, the beta coefficient for stock  $s$  is given by:

$$\beta_s = \frac{\sigma_{s,m}}{\sigma_m^2} \quad (6)$$

Where  $\sigma_{s,m}$  denotes the covariance of the return on stock  $s$  with the return on the broad market, and  $\sigma_m^2$  denotes the variance of the return on the broad market. For a single stock the SML equation is given by:

$$\mu_s = (\mu_m - r_f)\beta_s + r_f \quad (7)$$

And for a portfolio, it is given by:

$$\mu_p = (\mu_m - r_f)\beta_p + r_f \quad (8)$$



Contrary to conventional wisdom, a stock's beta is not, in fact, a measure of the stock's systematic risk in any *absolute* sense. Rather, it is a measure of the stock's systematic risk *relative* to the systematic risk of the broad market. Standard deviation, on the other hand, is an absolute measure of risk. For example, if I tell you that a stock's beta is 1.5 and that based on this beta the stock's expected return is 10%, would you consider that stock an attractive investment? The problem is that a beta of 1.5 would mean one thing if the broad market's standard deviation is 15% and something quite different if the broad market's standard deviation is 30%. Thus, standard deviation does count. Unfortunately, as already argued, the standard deviation of a single stock's return is a polluted measure of compensatory risk precisely because it includes an unsystematic component.

The upshot of this discussion is that it would likely be better, or at least useful, if we could isolate the systematic component of a stock's risk directly from the stock's standard deviation. Interestingly, this is rather easy to do. Indeed it can be derived several ways, always with the same result.<sup>xiii</sup> From this point forward, I will sometimes use the term "volatility" in lieu of the term "standard deviation of return" and I will occasionally make references to "vol points."<sup>xiv</sup> These terms are borrowed from the options literature. A standard deviation of return of 20% would be 20 vol points.

## **AN ALTERNATIVE MEASURE OF SYSTEMATIC RISK**

Denote for the moment a single stock or a portfolio of stocks by  $\lambda$ . I contend that the systematic risk component of a stock's (or a portfolio's) standard deviation is given by:

$$\text{Systematic risk} = \rho_{\lambda,m} \sigma_{\lambda} \quad (9)$$

and that this measure of systematic risk is simply Sharpe's beta measure scaled by the volatility of the broad market. That is  $\rho_{\lambda,m} \sigma_{\lambda} = \beta_{\lambda} \sigma_m$ .

And, since,

$$\text{Total risk } (\sigma_{\lambda}) = \text{Unsystematic Risk} + \text{Systematic Risk}$$

It follows that:

$$\sigma_{\lambda} = (1 - \rho_{\lambda,m}) \sigma_{\lambda} + \rho_{\lambda,m} \sigma_{\lambda} \quad (10)$$

Where  $\rho_{\lambda,m}$  denotes the correlation between the return associated with the stock or the portfolio, as appropriate, and the return on the broad market. Equation (10) is clearly a tautology.

If one chooses to measure risk using variance (Markowitz's original approach) instead of standard deviation, the relationship is as follows:

$$\begin{aligned} \sigma_{\lambda}^2 &= [(1 - \rho_{\lambda,m}) \sigma_{\lambda} + \rho_{\lambda,m} \sigma_{\lambda}]^2 \\ &= (1 - \rho_{\lambda,m}^2) \sigma_{\lambda}^2 + \rho_{\lambda,m}^2 \sigma_{\lambda}^2 \end{aligned} \quad (11)$$

If indeed the first term in Equation (10) (or the first term in Equation (11)) captures a stock's (or a portfolio's, as appropriate) unsystematic risk, then it should vanish as the portfolio is increasingly diversified. Consider now what happens to  $\rho_{\lambda,m}$  as the portfolio is increasingly

diversified. With increasing diversification, the number of stocks in the portfolio converges to the number of stocks in the broad market with the result that  $\rho_{\lambda,m} \rightarrow \rho_{m,m}$  as  $N \rightarrow M$ .<sup>xv</sup> Since, by definition, everything is perfectly correlated with itself,  $\rho_{m,m} = 1$  with the result that the first term on the right hand side of Equations (10) (and also in Equation (11)) goes to zero. When the portfolio's unsystematic risk is zero, the systematic risk is equal to that of the broad market. That is, the second term on the right becomes  $\sigma_m$  for Equation (10) and  $\sigma_m^2$  for Equations (11). Of course, the risk can be levered up (down) by appropriate borrowing (lending).

The question then is whether Equation (9) is indeed a valid measure of a stock's or a portfolio's systematic risk. I offer three different proofs, each of which offers different insights into this alternative measure of systematic risk.

### **PROOF 1: DERIVATION FROM A SINGLE-STOCK'S BETA**

Consider again the definition of a single-stock's beta from Equation (6), repeated below as Equation (12). Here  $s$  denotes a single stock and therefore  $\lambda = s$ .

$$\beta_s = \frac{\sigma_{s,m}}{\sigma_m^2} \tag{12}$$

Now consider the relationship between correlation and covariance:

$$\rho_{s,m} = \frac{\sigma_{s,m}}{\sigma_s \sigma_m} \tag{13}$$

implying that:  $\sigma_{s,m} = \rho_{s,m} \sigma_s \sigma_m$  (14)

Substituting Equation (14) into Equation (12), we obtain Equation (15).

$$\beta_s = \frac{\rho_{s,m} \sigma_s \sigma_m}{\sigma_m^2} \quad (15)$$

Which reduces to:

$$\beta_s = \frac{\rho_{s,m} \sigma_s}{\sigma_m} \quad (16)$$

or, equivalently:

$$\beta_s \sigma_m = \rho_{s,m} \sigma_s \quad (17)$$

Equation (17) proves the proposition.<sup>xvi</sup>

## **PROOF 2: DERIVATION FROM THE MARKET'S BETA**

Consider now a well-known property of beta. The beta of a portfolio, which is the systematic risk associated with the portfolio, is simply a weighted average of the betas of the portfolio's components:

$$\beta_p = \sum w_i \beta_i \quad (18)$$

As above (Equation (16)), this can be re-written as:

$$\beta_p = \sum w_i \frac{\rho_{i,m} \sigma_i}{\sigma_m} \quad (19)$$

Cross multiplying by  $\sigma_m$  we obtain,

$$\beta_p \sigma_m = \sum w_i \rho_{i,m} \sigma_i \quad (20)$$

Now consider the specific case when the portfolio is the market, that is  $\beta_p = \beta_m$ .

$$\beta_m \sigma_m = \sum w_i \rho_{i,m} \sigma_i \quad (21)$$

Since, by definition, the beta of the market is always 1, it follows that

$$\sigma_m = \sum w_i \rho_{i,m} \sigma_i \quad \text{when } N = M \quad (22)$$

And, since the standard deviation of the market as a whole is pure systematic risk, Equation (22) says that the systematic risk of the market is simply the weighted average of the systematic risks of the individual stocks that make up the market, and these are individually given by  $\rho_{i,m}\sigma_i$ .

### PROOF 3: RELATING THE CML TO THE SML

Using the same notation as above, the CML of Tobin was given earlier by Equation (5) and is repeated below as Equation (23):

$$\mu_p = \frac{(\mu_m - r_f)}{\sigma_m} \sigma_p + r_f \quad (23)$$

Equation (23) holds exactly whenever the portfolio is thoroughly diversified (so  $\sigma_p$  is purely systematic). Now allow for the possibility that the portfolio is not well diversified. Substitute  $\rho_{p,m}\sigma_p$  and  $\rho_{m,m}\sigma_m$  for  $\sigma_p$  and  $\sigma_m$ , respectively. Note that  $\rho_{m,m}\sigma_m = \sigma_m$  since everything is perfectly correlated with itself. Then Equation (23) becomes Equation (24), which I will refer to as the Adjusted CML:

$$\mu_p = \frac{(\mu_m - r_f)}{\sigma_m} \rho_{p,m}\sigma_p + r_f \quad (24)$$

or, equivalently,

$$= (\mu_m - r_f) \frac{\rho_{p,m} \sigma_p}{\sigma_m} + r_f \quad (25)$$

Since, as demonstrated earlier,

$$\beta_p = \frac{\rho_{p,m} \sigma_p}{\sigma_m}$$

Equation (25) reduces to:

$$\mu_p = (\mu_m - r_f) \beta_p + r_f \quad (26)$$

which is, of course, the SML version of CAPM and this holds whether the portfolio of interest is well diversified, not well diversified, or even a single stock.

## A SIMPLE DEMONSTRATION

As a simple demonstration, consider a stock whose standard deviation of return is 40%<sup>xvii</sup> and whose return correlation with the market proxy is 0.60. Assume that the standard deviation of the market's return is 15%,<sup>xviii</sup> that the expected market return is 8%, and that the relevant risk-free rate is 3%. Based on this information, the stock's beta is 1.60<sup>xix</sup> and the CAPM dictates an expected return for this stock of 11% [i.e.,  $\mu_s = (8\% - 3\%) \times 1.6 + 3\%$ ]. Thus, we can estimate the fair return dictated by the systematic component of this stock's risk, but we *cannot say without further effort* how much of the stock's volatility is systematic and how much is

unsystematic. The adjusted CML (but not the original CML) also dictates an expected return for this stock of 11% [i.e.,  $\mu_s = \left[ \frac{(8\% - 3\%)}{15\%} \times .6 \times 40\% \right] + 3\%$ ]. But it also tells us that the portion of the stock's volatility that is systematic is 60% (i.e.,  $\rho_{s,m} = 0.60$ ) of 40% or 24% and the portion of the stock's volatility that is unsystematic is 40% (i.e.,  $1 - \rho_{s,m} = 0.40$ ) of 40% or 16%. That is to say, of the 40 vol points that represent the standard deviation of the stock's return, 24 vol points represent systematic risk and 16 vol points represent unsystematic risk.

By decomposing a stock's (or a portfolio's) standard deviation into that portion that is systematic and that portion that is unsystematic we then have both a relative measure of systematic risk and an absolute measure of systematic risk, potentially facilitating the risk/reward tradeoff decision.

## **OTHER IMPLICATIONS**

There are several potentially useful implications of recognizing that the systematic component of a single stock's risk is given by  $\rho_{s,m} \sigma_s$ .

### **Reduction in Computational Complexity**

The approach to measuring systematic risk proposed here offers considerable computational advantages over Markowitz's original approach. Consider Markowitz's measure of portfolio variance given by Equation (3) and repeated below as Equation (27):



$$\sigma_p^2 = \sum_{i=1} w_i^2 \sigma_i^2 + \sum_{i \neq j} w_i w_j \sigma_i \sigma_j \rho_{i,j} \quad (27)$$

As already shown, for a thoroughly diversified portfolio, such as the entire market, this would then reduce to Equation (4) repeated below as Equation (28).

$$\sigma_m^2 = \sum_{i \neq j} w_i w_j \sigma_i \sigma_j \rho_{i,j} \quad (28)$$

If we choose to work in terms of standard deviation, rather than variance, this is given by:

$$\sigma_m = \sqrt{\sum_{i \neq j} w_i w_j \sigma_i \sigma_j \rho_{i,j}} \quad (29)$$

Even if we limit ourselves to a broad stock index like the S&P 500 as a proxy for the market, this measure of market risk requires the computation of an enormous number of cross correlations:

$$0.5 \times (500 \times 500 - 500) = 124,750.^{xx}$$

Now consider the approach proposed in this paper. Equation (22), derived earlier and repeated below as Equation (30), shows that the risk, as measured by the standard deviation of return, associated with the broad market is given by:

$$\sigma_m = \sum w_i \rho_{i,m} \sigma_i \quad (30)$$

For the market as a whole, these two measures (Equations (29) and (30)) of the market's standard deviation of return must produce identical results, since both are free of unsystematic risk. But Equation (30) requires the computation of far fewer correlations. In the case where we use the S&P 500 as a proxy for the market, Equation (30) requires the computation of 500 correlations, as opposed to the Markowitz measure given by Equation (29) that requires the computation of 124,750 correlations.

### Dispersion Trading

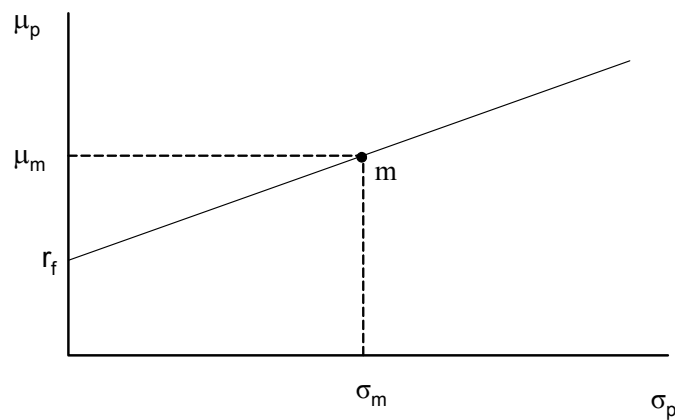
Dispersion trading is a sophisticated quantitative trading strategy most often practiced by volatility-focused hedge funds. It can be executed using either options or derivatives specifically structured to trade volatility, such as volatility swaps and variance swaps. In the option approach, the trader begins by computing the implied volatility of the index from options on the index (i.e., index options). Then, he separately computes the index volatility from the Markowitz formula using as inputs (1) the weights of the stocks currently included in the index, (2) the implied volatilities of all of the individual stocks in the index, as extracted from equity options on those stocks, and (3) the pair-wise return correlations<sup>xxi</sup> for all pairs of stocks included in the index. If the index-option's implied volatility exceeds the Markowitz volatility computed from the individual equity options' implied volatilities, the trader would short the index options and buy the equity options in the proportions that the stocks are represented in the index.<sup>xxii</sup> Of course, if the Markowitz volatility exceeded the index volatility implied by the index option, the strategy could be reversed.

As noted above, the number of cross-correlations that must be estimated to do this calculation correctly can be overwhelming. My proposed approach can achieve the same result with a small fraction of that number of computations.

### Improved Risk-Adjusted Performance Measurement

As noted earlier, the original CML of Tobin, if recast in terms of just the systematic component of risk, is identical to the SML of the CAPM. It is useful to show this graphically.

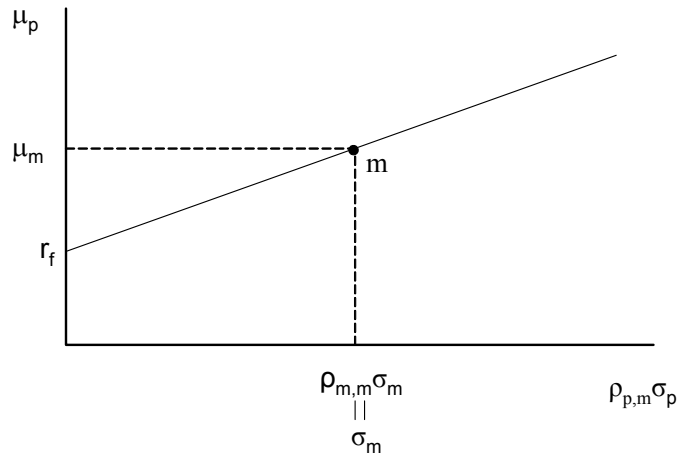
Begin with Tobin's original CML:



Original CML: Limited to well-diversified portfolios

If we now recast the model in terms of the systematic component of the standard deviation

$(\rho_{p,m}\sigma_p)$  we obtain an “adjusted CML”



Since  $\rho_{m,m} = 1$

Adjusted CML: Will work for any portfolio, even for a single stock

For the reasons detailed above, this will provide expected return forecasts identical to the SML. Note, of course, that  $\rho_{p,m}\sigma_p \leq \sigma_p$ , so that the adjusted CML will forecast a return lower than or equal to the original CML, just as the SML forecasts a return lower than or equal to the original CML.

If my proposed adjusted CML, is equivalent to the SML in terms of its single-stock and portfolio return forecasts, why bother?

There are several reasons we might prefer to derive return forecasts in terms of a correlation-adjusted standard deviation instead of beta. One application involves turning these *ex ante* models on their heads to derive *ex post* performance measures. For example, many practitioners prefer the Sharpe Ratio (SR) to the Treynor Index (TI) as a risk-adjusted performance metric. A key reason for this is that the standard deviation of return is not market specific. It is applicable to the stock market, the bond market, the alternatives market, and so on. Notwithstanding Sharpe's original intent, beta has always been associated with risk in the stock market and is therefore more limited in its applicability. But a key feature of the SR is that it

focuses on the total risk, even if the total risk is partially unsystematic in nature. For some purposes this is a strength of the SR and for other purposes it is a weakness. I propose an alternative measure of risk-adjusted returns which I call the “Author Ratio” (AR).<sup>xxiii</sup> Like the SR, the numerator of the AR is the “excess return” earned on the portfolio. But unlike the SR which uses standard deviation of return in the denominator, the AR limits the risk to the systematic component of the standard deviation.<sup>xxiv</sup> The AR has the benefit of the SR, which employs standard deviation, and the benefit of the TI, which limits risk to the systematic component. Note that the SR *for the market as a whole* and the AR *for the market as a whole* are identical. But for less than thoroughly diversified portfolios they will be different. Also notice that because the denominator of the AR can be expressed as the beta of the portfolio times the volatility of the market (Equation (17)) and the volatility of the market can be viewed as a constant across all portfolios drawn from the same asset class, the AR will always produce the same rankings as the TI when their application is limited to a single asset class.

Sharpe Ratio (SR):

$$SR = \frac{r_p - r_f}{\sigma_p}$$

Author Ratio (AR):

$$AR = \frac{r_p - r_f}{\rho_{p,m} \sigma_p}$$

Treynor Index (TI)

$$TI = \frac{r_p - r_f}{\beta_p}$$

## CONCLUSION

I have demonstrated that it is possible to decompose a single stock’s risk, when measured by either standard deviation or by variance, into that portion that is unsystematic (and therefore not relevant to estimating expected return), and that portion that is systematic (and therefore relevant to estimating expected return). This decomposition provides an absolute measure of

systematic risk, as opposed to the traditional beta measure which is a relative measure of systematic risk. The proposed measures are completely consistent with Markowitz's measure of portfolio risk, Tobin's Capital Market Line, and Sharpe's Security Market Line. In some situations, the approach proposed here to decompose risk into its two fundamental parts might offer computational advantages similar to those offered by the use of beta, but allowing one to stay in the framework of mean/standard deviation space. This has implications for some volatility-based trading strategies. The proposed approach also offers a new way to look at risk-adjusted returns which offers the benefits of both the Sharpe Ratio and the Treynor Index.

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- <sup>i</sup> This is evidenced by the adage “don’t put all of your eggs in one basket” which dates from at least 1756.
- <sup>ii</sup> To be applicable without modification, the profit/loss functions associated with the assets must be linear and the return distributions must be normal.
- <sup>iii</sup> The minimum-variance set is the set of portfolios such that, for any given level of expected return, the variance (and by implication, the standard deviation) is minimized. The efficient frontier is that portion of the minimum-variance set for which expected return is maximized for any given variance.
- <sup>iv</sup> The “degree of diversification” is a function of both the number of stocks included in the portfolio and how the value of the portfolio is divided among them (i.e., the weighting scheme). Generally, to be well-diversified a portfolio must contain many stocks and the weights allocated to each must be small.
- <sup>v</sup> Unsystematic risk is also known as idiosyncratic risk and as company specific risk.
- <sup>vi</sup> Systematic risk is also known as market risk since it is that portion of portfolio risk explained by the movements of the broad market.
- <sup>vii</sup> Technically, for this result to hold when leverage is applied, the investor must be able to both borrow and lend at the risk-free rate. To the degree that the borrowing and lending rates differ, there will be a “kink” in the otherwise linear efficient frontier.
- <sup>viii</sup> The phrase “thoroughly diversified” implies that the degree of diversification is sufficient to virtually eliminate unsystematic risk.
- <sup>ix</sup> Because unsystematic risk will diversify away for all investors, it was soon realized that investors should only be compensated for the systematic risk that they bear. That is, in an efficient market, an asset’s expected return should be directly related to the asset’s degree of systematic risk and unrelated to the asset’s unsystematic risk.
- <sup>x</sup> Other contributors to this work included Treynor (1962), Lintner (1965) and Mossin (1966).
- <sup>xi</sup> Sharpe (1963).
- <sup>xii</sup> Reilly and Wright (1988) found that using monthly as opposed to weekly data is a cause for differences in betas, but the effect is diminished as the size of the firm increases.
- <sup>xiii</sup> Despite the ease with which the result in the next section can be derived, it does not seem to have been emphasized anywhere in the literature.
- <sup>xiv</sup> Technically, a stock’s volatility is the standard deviation of the annual percentage price change measured on a continuously compounded basis. In order to use volatility in lieu of standard deviation we must assume that the stock is non-dividend paying so that “price return” and “total return” are identical.
- <sup>xv</sup> This statement assumes that the weighting scheme used in the portfolio will converge to the weighting scheme employed in the broad market (as defined by the market proxy).
- <sup>xvi</sup> Note that the standard deviation of the market is purely systematic risk and may, in this context, be taken as a constant in the sense that, at a given moment in time, it is the same irrespective of which particular stock one is looking at.
- <sup>xvii</sup> This can be calculated from the stock’s recent historic returns, or it can be taken as the implied volatility of return as extracted from equity options on the stock (the latter approach implicitly assumes that the stock is not dividend paying so that price return and total return are equivalent).
- <sup>xviii</sup> Again, this can be derived from recent historic market data or extracted from index options.
- <sup>xix</sup> Computed from raw returns.
- <sup>xx</sup> The correlation matrix would consist of  $500 \times 500$  correlations. However, we are only concerned with the cross correlations and can therefore deduct the 500 that lie on the principal diagonal. This leaves 249,500. Further, because the matrix is symmetric, this number is reduced by half.
- <sup>xxi</sup> These pair-wise return correlations can be computed from recent historic returns or they can be instantaneous correlations derived in other ways.
- <sup>xxii</sup> This is an oversimplification. An option on a portfolio is not equivalent to a portfolio of options, necessitating continuous rebalancing to maintain equivalence. For a more thorough discussion of dispersion trading in the context of the risk measure proposed in this paper, see Marshall (2008).
- <sup>xxiii</sup> I would have preferred to call this either the “Adjusted Sharpe Ratio” or the “Modified Sharpe Ratio” but both of those terms are already in use for other purposes. The “Adjusted Sharpe Ratio,” attributable to Johnson et al. (2002) is defined as the Sharpe Ratio that would be implied by the “downside deviation if returns were distributed normally.” The term “Modified Sharpe Ratio” is often used to mean the ratio of a portfolio’s excess return to its modified value-at-risk and has been employed in the “alternative investments” sphere. Both terms have also been used in other contexts.

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<sup>xxiv</sup> A different measure of risk-adjusted performance sometimes used in the alternative investments literature that also considers correlation of return with the market is the BAVAR (Beta And Volatility Adjusted Return) Ratio. This is discussed in Horowitz (2004, p.257).



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## APPENDIX 1: Proof of the Basic Markowitz Equation

By expectations definition of variance:

$$\sigma_p^2 = E[(r_p - \mu_p)^2] \quad (\text{A1.1})$$

By the definition of  $r_p$  (portfolio return) and  $\mu_p$  (portfolio mean return):

$$= E\left[\left(\sum_{i=1}^N w_i r_i - \sum_{i=1}^N w_i \mu_i\right)^2\right] \quad (\text{A1.2})$$

Combining terms:

$$= E\left[\left(\sum_{i=1}^N w_i (r_i - \mu_i)\right)^2\right] \quad (\text{A1.3})$$

Expanding terms:

$$= E\left[\sum_{i=1}^N \sum_{j=1}^N w_i w_j (r_i - \mu_i)(r_j - \mu_j)\right] \quad (\text{A1.4})$$

By the linearity property of the expectations operator:

$$= \sum_{i=1}^N \sum_{j=1}^N w_i w_j E[(r_i - \mu_i)(r_j - \mu_j)] \quad (\text{A1.5})$$

By expectations definition of covariance:

$$= \sum_{i=1}^N \sum_{j=1}^N w_i w_j \sigma_{i,j} \quad (\text{A1.6})$$

Which completes the proof.

## APPENDIX 2: Proof that Unsystematic Risk Diversifies Away

$$\text{Unsystematic Risk} = \sum_{i=1}^N w_i^2 \sigma_i^2 \quad (\text{A2.1})$$

For ease of illustration, suppose each weight is as small as possible requiring that  $w_i = \frac{1}{N}$  for all

i. We can then re-write Equation (A2.1) as:

$$\sum_{i=1}^N \left(\frac{1}{N}\right)^2 \sigma_i^2 \quad (\text{A2.2})$$

or

$$\left(\frac{1}{N}\right)^2 \sum_{i=1}^N \sigma_i^2 \quad (\text{A2.3})$$

Recognizing that  $\sum_{i=1}^N \sigma_i^2 = N\sigma_{avg}^2$ , where  $\sigma_{avg}^2$  denotes the average variance of the stocks

included in the portfolio, we can re-write Equation (A2.3) as:

$$= \left(\frac{1}{N}\right)^2 \times N\sigma_{avg}^2 \quad (\text{A2.4})$$

or

$$= \frac{\sigma_{avg}^2}{N} \quad (\text{A2-5})$$

Recognizing that individual stock's variances are finite and therefore average variance is finite, it is readily obvious that Equation (A2.5) goes to zero as N grows larger.

**APPENDIX 3:** Proof that beta computed from excess returns equals beta computed from raw returns when the risk-free rate is constant.

Beginning with stock  $i$ 's beta defined as the ratio of (1) the covariance of the excess returns of stock  $i$  and the market's excess return, and (2) the variance of the market's excess return:

$$\beta_i = \frac{\sigma_{i-f,m-f}}{\sigma_{m-f}^2} \quad (\text{A3.1})$$

Where  $\sigma_{i-f,m-f}$  denotes the covariance of the excess return on stock  $i$  and the excess return on the market, and  $\sigma_{m-f}^2$  denotes the variance of the excess return on the market.

Re-expressing this in expectations notation we have:

$$\beta_i = \frac{E[((r_i - r_f) - (\mu_i - \mu_f))(r_m - r_f) - (\mu_m - \mu_f)]}{E[(r_m - r_f) - (\mu_m - \mu_f)]^2} \quad (\text{A3.2})$$

If the risk-free rate is constant (as it is in all ex ante applications), then  $r_f = \mu_f$  and the equation above reduces to:

$$\beta_i = \frac{E[(r_i - \mu_i)(r_m - \mu_m)]}{E[(r_m - \mu_m)^2]} \quad (\text{A3.3})$$

Which is simply the beta computed with respect to raw returns rather than with respect to excess returns.

$$\beta_i = \frac{\sigma_{i,m}}{\sigma_m^2} \quad (\text{A3.4})$$